

## CONGRUENCES CONCERNING LEGENDRE POLYNOMIALS II

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ABSTRACT. Let  $p > 3$  be a prime, and let  $m$  be an integer with  $p \nmid m$ . In the paper we solve some conjectures of Z.W. Sun concerning  $\sum_{k=0}^{p-1} \binom{2k}{k}^3 / m^k \pmod{p^2}$ ,  $\sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} / m^k \pmod{p}$  and  $\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} / m^k \pmod{p^2}$ . In particular, we show that  $\sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^3 \equiv 0 \pmod{p^2}$  for  $p \equiv 3, 5, 6 \pmod{7}$ . Let  $P_n(x)$  be the Legendre polynomials. In the paper we also show that  $P_{[\frac{p}{4}]}(t) \equiv -(\frac{-6}{p}) \sum_{x=0}^{p-1} (\frac{x^3 - \frac{3}{2}(3t+5)x - 9t-7}{p}) \pmod{p}$  and determine  $P_{\frac{p-1}{2}}(\sqrt{2}), P_{\frac{p-1}{2}}(\frac{3\sqrt{2}}{4}), P_{\frac{p-1}{2}}(\sqrt{-3}), P_{\frac{p-1}{2}}(\frac{\sqrt{3}}{2}), P_{\frac{p-1}{2}}(\sqrt{-63}), P_{\frac{p-1}{2}}(\frac{3\sqrt{7}}{8}) \pmod{p}$ , where  $t$  is a rational  $p$ -integer,  $[x]$  is the greatest integer not exceeding  $x$  and  $(\frac{a}{p})$  is the Legendre symbol. As consequences we determine  $P_{[\frac{p}{4}]}(t) \pmod{p}$  in the cases  $t = -\frac{5}{3}, -\frac{7}{9}, -\frac{65}{63}$  and confirm many conjectures of Z.W. Sun.

MSC: Primary 11A07, Secondary 33C45, 11E25, 11L10, 05A10, 05A19

Keywords: Legendre polynomial; congruence; character sum; binary quadratic form

**1. Introduction.**Let  $p$  be an odd prime. Following [A] we define

$$A(p, \lambda) = \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 \binom{\frac{p-1}{2} + k}{2k} \lambda^{kp}.$$

It is easily seen ([S3, Lemmas 2.2 and 2.4]) that

$$(1.1) \quad \binom{\frac{p-1}{2}}{k} \equiv \frac{\binom{2k}{k}}{(-4)^k} \pmod{p} \quad \text{and} \quad \binom{\frac{p-1}{2} + k}{2k} \equiv \frac{\binom{2k}{k}}{(-16)^k} \pmod{p^2}.$$

Thus, by Fermat's little theorem, for any rational  $p$ -integer  $\lambda$ ,

$$(1.2) \quad A(p, \lambda) \equiv \sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^3 \left(\frac{\lambda}{64}\right)^k \pmod{p}.$$

For positive integers  $a, b$  and  $n$ , if  $n = ax^2 + by^2$  for some integers  $x$  and  $y$ , we briefly say that  $n = ax^2 + by^2$ . In 1985, Beukers[B] conjectured a congruence for  $A(p, 1) \pmod{p^2}$  equivalent to

$$(1.3) \quad \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{64^k} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \\ 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 4y^2 \equiv 1 \pmod{4}. \end{cases}$$

This congruence was proved by several authors including Ishikawa[Is] ( $p \equiv 1 \pmod{4}$ ), van Hamme[vH] ( $p \equiv 3 \pmod{4}$ ) and Ahlgren[A]. In 1998, by using the hypergeometric series  ${}_3F_2(\lambda)_p$  over the finite field  $F_p$ , Ono[O] obtained congruences for  $A(p, \lambda) \pmod{p}$  in the cases  $\lambda = -1, -8, -\frac{1}{8}, 4, \frac{1}{4}, 64, \frac{1}{64}$ , see also [A]. Hence from (1.2) we deduce congruences for  $\sum_{k=0}^{\frac{p-1}{2}} \frac{1}{m^k} \binom{2k}{k}^3 \pmod{p}$  in the cases  $m = 1, -8, -64, 256, -512, 4096$ . Recently the author's brother Zhi-Wei Sun[Su1, Su3, Su4] conjectured corresponding congruences modulo  $p^2$ . In particular, he conjectured that

$$(1.4) \quad \sum_{k=0}^{p-1} \binom{2k}{k}^3 \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod{7}, \\ 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 7y^2 \equiv 1, 2, 4 \pmod{7}. \end{cases}$$

We note that  $p \mid \binom{2k}{k}$  for  $\frac{p+1}{2} \leq k \leq p-1$ . In the paper, by using Legendre polynomials we partially solve Zhi-Wei Sun's such conjectures. For example, we prove (1.4) in the case  $p \equiv 3, 5, 6 \pmod{7}$ .

Let  $\{P_n(x)\}$  be the Legendre polynomials given by

$$P_0(x) = 1, \quad P_1(x) = x, \quad (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \quad (n \geq 1).$$

It is well known that (see [MOS, pp. 228-232], [G, (3.132)-(3.133)])

$$(1.5) \quad P_n(x) = \frac{1}{2^n} \sum_{k=0}^{[n/2]} \binom{n}{k} (-1)^k \binom{2n-2k}{n} x^{n-2k} = \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n} (x^2 - 1)^n,$$

where  $[x]$  is the greatest integer not exceeding  $x$ . From (1.5) we see that

$$(1.6) \quad P_n(-x) = (-1)^n P_n(x), \quad P_{2m+1}(0) = 0 \quad \text{and} \quad P_{2m}(0) = \frac{(-1)^m}{2^{2m}} \binom{2m}{m}.$$

In the paper we deduce our main results for congruences from the identity  $\sum_{k=0}^n \binom{2k}{k}^2 \binom{n+k}{2k} x^k = P_n(\sqrt{1+4x})^2$ .

Let  $\mathbb{Z}$  be the set of integers. For a prime  $p$  let  $\mathbb{Z}_p$  be the set of rational numbers whose denominator is coprime to  $p$ , and let  $\left(\frac{a}{p}\right)$  be the Legendre symbol. On the basis of the work of Ono[O] (see also [A, Theorem 2] and [LR]), in [S3, Theorem 2.11] the author showed that for a prime  $p > 3$  and  $t \in \mathbb{Z}_p$ ,

$$(1.7) \quad P_{\frac{p-1}{2}}(t) \equiv -\left(\frac{-6}{p}\right) \sum_{x=0}^{p-1} \left( \frac{x^3 - 3(t^2 + 3)x + 2t(t^2 - 9)}{p} \right) \pmod{p}.$$

In the paper, using (1.7) we prove that

$$(1.8) \quad P_{[\frac{p}{4}]}(t) \equiv -\left(\frac{6}{p}\right) \sum_{x=0}^{p-1} \left( \frac{x^3 - \frac{3}{2}(3t+5)x + 9t+7}{p} \right) \pmod{p}.$$

As consequences of (1.8), we determine  $P_{[\frac{p}{4}]}(-\frac{5}{3})$ ,  $P_{[\frac{p}{4}]}(-\frac{7}{9})$ ,  $P_{[\frac{p}{4}]}(-\frac{65}{63}) \pmod{p}$  and use them to solve Z.W. Sun's conjectures on  $\sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} / m^k \pmod{p}$  and  $\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} / m^k \pmod{p^2}$ , see [Su1]. For instance, for any prime  $p > 7$ ,

$$(1.9) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{81^k} \equiv \begin{cases} 4C^2 \pmod{p} & \text{if } p \equiv 1, 2, 4 \pmod{7} \text{ and so } p = C^2 + 7D^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases}$$

## 2. Congruences for $P_{\frac{p-1}{2}}(\sqrt{x})$ and $P_{[\frac{p}{4}]}(t) \pmod{p}$ .

**Lemma 2.1.** *Let  $p$  be an odd prime and  $k \in \{0, 1, \dots, [\frac{p}{4}]\}$ . Then*

$$\binom{[\frac{p}{4}] + k}{2k} \equiv \frac{1}{(-64)^k} \binom{4k}{2k} \pmod{p} \text{ and } \binom{p-1-2k}{\frac{p-1}{2}} \equiv \frac{(-1)^{\frac{p-1}{2}}}{16^k} \binom{4k}{2k} \pmod{p}.$$

Proof. Suppose  $r = 1$  or  $3$  according as  $4 \mid p-1$  or  $4 \mid p-3$ . Then clearly

$$\begin{aligned} \binom{[\frac{p}{4}] + k}{2k} &= \frac{(\frac{p-r}{4} + k)(\frac{p-r}{4} + k - 1) \cdots (\frac{p-r}{4} - k + 1)}{(2k)!} \\ &\equiv (-1)^k \frac{(4k-r)(4k-r-4) \cdots (4-r) \cdot r(r+4) \cdots (4k+r-4)}{4^{2k} \cdot (2k)!} \\ &= \frac{(-1)^k \cdot (4k)!}{2^{2k} \cdot (2k)! \cdot 4^{2k} \cdot (2k)!} = \frac{\binom{4k}{2k}}{(-64)^k} \pmod{p} \end{aligned}$$

and

$$\begin{aligned} &(-1)^{\frac{p-1}{2}} \binom{p-1-2k}{\frac{p-1}{2}} \\ &= (-1)^{\frac{p-1}{2}} \frac{(p-1-2k)(p-2-2k) \cdots (p - (\frac{p-1}{2} + 2k))}{\frac{p-1}{2}!} \\ &\equiv \frac{(2k+1)(2k+2) \cdots (\frac{p-1}{2} + 2k)}{\frac{p-1}{2}!} = \frac{(\frac{p-1}{2} + 2k)(\frac{p-1}{2} + 2k - 1) \cdots (\frac{p-1}{2} + 1)}{(2k)!} \\ &\equiv \frac{(4k-1)(4k-3) \cdots 3 \cdot 1}{2^{2k} \cdot (2k)!} = \frac{(4k)!}{2^{2k} \cdot (2k)! \cdot 2^{2k} \cdot (2k)!} = \frac{1}{2^{4k}} \binom{4k}{2k} \pmod{p}. \end{aligned}$$

This proves the lemma.

**Lemma 2.2.** *Let  $p$  be an odd prime and let  $t$  be a variable. Then*

$$P_{[\frac{p}{4}]}(t) \equiv \sum_{k=0}^{[p/4]} \binom{4k}{2k} \binom{2k}{k} \left(\frac{1-t}{128}\right)^k \pmod{p}.$$

Proof. It is known that  $([G, (3.135)])$

$$(2.1) \quad P_n(t) = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \left(\frac{t-1}{2}\right)^k.$$

Observe that  $\binom{n}{k} \binom{n+k}{k} = \binom{n+k}{2k} \binom{2k}{k}$ . By (2.1) and Lemma 2.1 we have

$$P_{[\frac{p}{4}]}(t) = \sum_{k=0}^{[p/4]} \binom{[\frac{p}{4}] + k}{2k} \binom{2k}{k} \left(\frac{t-1}{2}\right)^k \equiv \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k} \binom{2k}{k}}{(-64)^k} \left(\frac{t-1}{2}\right)^k \pmod{p}.$$

This yields the result.

**Lemma 2.3.** *Let  $p$  be an odd prime and let  $t$  be a variable. Then*

$$\sum_{k=0}^{[p/4]} \binom{4k}{2k} \binom{2k}{k} \left(\frac{1-t}{128}\right)^k \equiv (-1)^{[\frac{p}{4}]} \sum_{k=0}^{[p/4]} \binom{4k}{2k} \binom{2k}{k} \left(\frac{1+t}{128}\right)^k \pmod{p}.$$

Proof. This is immediate from Lemma 2.2 and (1.6).

**Lemma 2.4.** *Let  $p$  be an odd prime and let  $x$  be a variable. Then*

$$(\sqrt{x})^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{x}) \equiv (-1)^{[\frac{p}{4}]} \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k} \binom{2k}{k}}{64^k} x^{\frac{p-1}{2}-k} \pmod{p}.$$

Proof. From (1.5) we see that

$$(\sqrt{x})^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{x}) = \frac{1}{2^{\frac{p-1}{2}}} \sum_{k=0}^{[p/4]} \binom{\frac{p-1}{2}}{k} \binom{p-1-2k}{\frac{p-1}{2}} (-1)^k x^{\frac{p-1}{2}-k}.$$

Thus applying (1.1) and Lemma 2.1 we obtain

$$(\sqrt{x})^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{x}) \equiv \left(\frac{-2}{p}\right) \sum_{k=0}^{[p/4]} \frac{1}{(-4)^k} \binom{2k}{k} \cdot \frac{1}{2^{4k}} \binom{4k}{2k} (-1)^k x^{\frac{p-1}{2}-k} \pmod{p}.$$

Noting that  $\left(\frac{-2}{p}\right) = (-1)^{[\frac{p}{4}]}$  we then obtain the result.

**Lemma 2.5.** *Let  $p$  be an odd prime and let  $x \in \mathbb{Z}_p$  with  $x \not\equiv 0, 1 \pmod{p}$ . Then*

$$\begin{aligned} & \left( \frac{2(1-x)}{p} \right) \left( \sqrt{\frac{x}{x-1}} \right)^{\frac{p-1}{2}} P_{\frac{p-1}{2}} \left( \sqrt{\frac{x}{x-1}} \right) \\ & \equiv (\sqrt{x})^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{x}) \equiv \left( \frac{x}{p} \right) P_{[\frac{p}{4}]} \left( \frac{2}{x} - 1 \right) \pmod{p}. \end{aligned}$$

Proof. By (1.6), Lemmas 2.2 and 2.4 we have

$$\begin{aligned} P_{[\frac{p}{4}]} \left( \frac{2}{x} - 1 \right) &= (-1)^{[\frac{p}{4}]} P_{[\frac{p}{4}]} \left( 1 - \frac{2}{x} \right) \equiv (-1)^{[\frac{p}{4}]} \sum_{k=0}^{[p/4]} \binom{4k}{2k} \binom{2k}{k} \frac{1}{(64x)^k} \\ &\equiv \left( \frac{x}{p} \right) (\sqrt{x})^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{x}) \pmod{p}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \left( \sqrt{\frac{x}{x-1}} \right)^{\frac{p-1}{2}} P_{\frac{p-1}{2}} \left( \sqrt{\frac{x}{x-1}} \right) \\ & \equiv \left( \frac{x/(x-1)}{p} \right) P_{[\frac{p}{4}]} \left( \frac{2}{x/(x-1)} - 1 \right) = \left( \frac{x(x-1)}{p} \right) P_{[\frac{p}{4}]} \left( 1 - \frac{2}{x} \right) \\ & \equiv \left( \frac{-2(x-1)}{p} \right) (\sqrt{x})^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{x}) \pmod{p}. \end{aligned}$$

This completes the proof.

**Lemma 2.6.** *Let  $p > 3$  be a prime and  $u \in \mathbb{Z}_p$ . Then*

$$(\sqrt{u})^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{u}) \equiv - \left( \frac{-6}{p} \right) \sum_{x=0}^{p-1} \left( \frac{ux^3 - 3(u+3)x + 2(u-9)}{p} \right) \pmod{p}.$$

Proof. For  $t \in \mathbb{Z}_p$  with  $t \not\equiv 0 \pmod{p}$ , by (1.7) we have

$$\begin{aligned} P_{\frac{p-1}{2}}(t) &\equiv - \left( \frac{-6}{p} \right) \sum_{x=0}^{p-1} \left( \frac{x^3 - 3(t^2+3)x + 2t(t^2-9)}{p} \right) \\ &= - \left( \frac{-6}{p} \right) \sum_{x=0}^{p-1} \left( \frac{(tx)^3 - 3(t^2+3)tx + 2t(t^2-9)}{p} \right) \\ &= - \left( \frac{-6t}{p} \right) \sum_{x=0}^{p-1} \left( \frac{t^2x^3 - 3(t^2+3)x + 2(t^2-9)}{p} \right) \pmod{p}. \end{aligned}$$

Hence, if  $u = t^2$  for some  $t \in \mathbb{Z}_p$  with  $t \not\equiv 0 \pmod{p}$ , then

$$\begin{aligned} (2\sqrt{u})^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{u}) &= (2t)^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(t) \equiv - \left( \frac{-3}{p} \right) \sum_{x=0}^{p-1} \left( \frac{ux^3 - 3(u+3)x + 2(u-9)}{p} \right) \\ &\equiv - \left( \frac{-3}{p} \right) \sum_{x=0}^{p-1} ((x^3 - 3x + 2)u - 9x - 18)^{\frac{p-1}{2}} \pmod{p}. \end{aligned}$$

This is also true for  $u = 0$  since  $\sum_{x=0}^{p-1} \left(\frac{-9x-18}{p}\right) = 0$ . Set

$$f(u) = (2\sqrt{u})^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{u}) + \left(\frac{-3}{p}\right) \sum_{x=0}^{p-1} ((x^3 - 3x + 2)u - 9x - 18)^{\frac{p-1}{2}}.$$

Then  $f(u) \equiv 0 \pmod{p}$  for  $u = 0^2, 1^2, 2^2, \dots, (\frac{p-1}{2})^2$ . From the proof of Lemma 2.4 we know that  $(2\sqrt{u})^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{u})$  is a polynomial of  $u$  with degree  $\frac{p-1}{2}$  and integral coefficients. Thus  $f(u)$  is a polynomial of  $u$  with degree at most  $\frac{p-1}{2}$  and integral coefficients. As  $f(u) \equiv 0 \pmod{p}$  for  $u = 0^2, 1^2, 2^2, \dots, (\frac{p-1}{2})^2$ , using Lagrange's theorem we see that all the coefficients in  $f(u)$  are divisible by  $p$ . Therefore,  $f(u) \equiv 0 \pmod{p}$  for every  $u \in \mathbb{Z}_p$ . This yields the result.

Now we are ready to prove the following main result.

**Theorem 2.1.** *Let  $p$  be a prime greater than 3 and  $t \in \mathbb{Z}_p$ .*

(i) *If  $t \not\equiv 0 \pmod{p}$ , then*

$$\begin{aligned} (\sqrt{t})^{-\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{t}) &\equiv P_{[\frac{p}{4}]} \left( \frac{2-t}{t} \right) \equiv \sum_{k=0}^{[p/4]} \binom{4k}{2k} \binom{2k}{k} \left( \frac{t-1}{64t} \right)^k \\ &\equiv (-1)^{[p/4]} \sum_{k=0}^{[p/4]} \binom{4k}{2k} \binom{2k}{k} \frac{1}{(64t)^k} \\ &\equiv - \left( \frac{-6t}{p} \right) \sum_{x=0}^{p-1} \left( \frac{tx^3 - 3(t+3)x + 2(t-9)}{p} \right) \pmod{p}. \end{aligned}$$

(ii) *We have*

$$P_{[\frac{p}{4}]}(t) \equiv \sum_{k=0}^{[p/4]} \binom{4k}{2k} \binom{2k}{k} \left( \frac{1-t}{128} \right)^k \equiv - \left( \frac{6}{p} \right) \sum_{x=0}^{p-1} \left( \frac{x^3 - \frac{3(3t+5)}{2}x + 9t + 7}{p} \right) \pmod{p}.$$

Proof. By Lemmas 2.5 and 2.6 we have

$$\begin{aligned} P_{[\frac{p}{4}]}(2/t - 1) \\ \equiv \left( \frac{t}{p} \right) (\sqrt{t})^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{t}) \equiv - \left( \frac{-6t}{p} \right) \sum_{x=0}^{p-1} \left( \frac{tx^3 - 3(t+3)x + 2(t-9)}{p} \right) \pmod{p}. \end{aligned}$$

This together with Lemmas 2.2 and 2.3 gives the first part. Substituting  $t$  by  $\frac{2}{t+1}$  and noting that

$$\begin{aligned} \sum_{x=0}^{p-1} \left( \frac{x^3 - \frac{3(3t+5)}{2}x - 9t - 7}{p} \right) &= \sum_{x=0}^{p-1} \left( \frac{(-x)^3 - \frac{3(3t+5)}{2}(-x) - 9t - 7}{p} \right) \\ &= \left( \frac{-1}{p} \right) \sum_{x=0}^{p-1} \left( \frac{x^3 - \frac{3(3t+5)}{2}x + 9t + 7}{p} \right) \end{aligned}$$

we then obtain the second part in the case  $t \not\equiv -1 \pmod{p}$ . When  $t \equiv -1 \pmod{p}$  we have  $P_{[\frac{p}{4}]}(-1) = (-1)^{[\frac{p}{4}]} = (\frac{-2}{p})$  and

$$\sum_{x=0}^{p-1} \left( \frac{x^3 - 3x - 2}{p} \right) = \sum_{x=0}^{p-1} \left( \frac{(x+1)^2(x-2)}{p} \right) = \sum_{x=0}^{p-1} \left( \frac{x-2}{p} \right) - \left( \frac{-1-2}{p} \right) = -\left( \frac{-3}{p} \right).$$

Thus the second part is also true for  $t \equiv -1 \pmod{p}$ . The proof is now complete.

**Corollary 2.1.** *Let  $p \geq 17$  be a prime and  $t \in \mathbb{Z}_p$ . Then*

$$\sum_{x=0}^{p-1} \left( \frac{x^3 - \frac{3(3t+5)}{2}x + 9t + 7}{p} \right) = \left( \frac{2}{p} \right) \sum_{x=0}^{p-1} \left( \frac{x^3 + \frac{3(3t-5)}{2}x + 9t - 7}{p} \right).$$

Proof. Since  $P_{[\frac{p}{4}]}(t) = (-1)^{[\frac{p}{4}]} P_{[\frac{p}{4}]}(-t)$ , by Theorem 2.3(ii) we obtain

$$\begin{aligned} & \sum_{x=0}^{p-1} \left( \frac{x^3 - \frac{3(3t+5)}{2}x + 9t + 7}{p} \right) \\ & \equiv (-1)^{[\frac{p}{4}]} \sum_{x=0}^{p-1} \left( \frac{x^3 - \frac{3(-3t+5)}{2}x - 9t + 7}{p} \right) = \left( \frac{-2}{p} \right) \sum_{x=0}^{p-1} \left( \frac{(-x)^3 + \frac{3(3t-5)}{2}(-x) - 9t + 7}{p} \right) \\ & = \left( \frac{2}{p} \right) \sum_{x=0}^{p-1} \left( \frac{x^3 + \frac{3(3t-5)}{2}x + 9t - 7}{p} \right) \pmod{p}. \end{aligned}$$

By Weil's estimate ([BEW, p.183]) we have

$$\left| \sum_{x=0}^{p-1} \left( \frac{x^3 - \frac{3(3t+5)}{2}x + 9t + 7}{p} \right) \right| \leq 2\sqrt{p} \quad \text{and} \quad \left| \sum_{x=0}^{p-1} \left( \frac{x^3 + \frac{3(3t-5)}{2}x + 9t - 7}{p} \right) \right| \leq 2\sqrt{p}.$$

Since  $4\sqrt{p} < p$  for  $p \geq 17$ , from the above we deduce the result.

For any prime  $p > 3$ , in [Su1] Zhi-Wei Sun conjectured congruences for  $\sum_{k=0}^{p-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{m^k} \pmod{p^2}$  in the cases  $m = 48, 63, 72$ . Now we confirm his congruences modulo  $p$ .

**Theorem 2.2.** *Let  $p$  be a prime greater than 3. Then*

$$\begin{aligned} P_{[\frac{p}{4}]} \left( -\frac{7}{9} \right) & \equiv \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k} \binom{2k}{k}}{72^k} \equiv (-1)^{[\frac{p}{4}]} \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k} \binom{2k}{k}}{576^k} \\ & \equiv \begin{cases} (-1)^{\frac{p-1}{4}} \left( \frac{p}{3} \right) 2a \pmod{p} & \text{if } 4 \mid p-1, p = a^2 + b^2 \text{ and } 4 \mid a-1, \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

Proof. From [BEW, Theorem 6.2.9] or [S1, (2.15)-(2.16)] we have

$$(2.2) \quad \sum_{x=0}^{p-1} \left( \frac{x^3 - 4x}{p} \right) = \begin{cases} -2a & \text{if } p \equiv 1 \pmod{4}, p = a^2 + 4b^2 \text{ and } a \equiv 1 \pmod{4}, \\ 0 & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Thus taking  $t = 9$  in Theorem 2.1(i) we deduce the result.

**Theorem 2.3.** *Let  $p$  be a prime greater than 3. Then*

$$\begin{aligned} P_{[\frac{p}{4}]} \left( -\frac{5}{3} \right) &\equiv \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k} \binom{2k}{k}}{48^k} \equiv (-1)^{[\frac{p}{4}]} \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k} \binom{2k}{k}}{(-192)^k} \\ &\equiv \begin{cases} 2A \pmod{p} & \text{if } 3 \mid p-1, p = A^2 + 3B^2 \text{ and } 3 \mid A-1, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

Proof. It is known that (see for example [S1, (2.7)-(2.9)] or [BEW, pp. 195-196])

$$(2.3) \quad \sum_{x=0}^{p-1} \left( \frac{x^3 + 8}{p} \right) = \begin{cases} -2A \left( \frac{2}{p} \right) & \text{if } 3 \mid p-1, p = A^2 + 3B^2 \text{ and } 3 \mid A-1, \\ 0 & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Thus, putting  $t = -3$  in Theorem 2.1(i) we deduce the result.

**Theorem 2.4.** *Let  $p \neq 2, 3, 7$  be a prime. Then*

$$\begin{aligned} P_{[\frac{p}{4}]} \left( -\frac{65}{63} \right) &\equiv \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k} \binom{2k}{k}}{63^k} \equiv (-1)^{[\frac{p}{4}]} \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k} \binom{2k}{k}}{(-4032)^k} \\ &\equiv \begin{cases} 2C \left( \frac{p}{3} \right) \left( \frac{C}{7} \right) \pmod{p} & \text{if } p \equiv 1, 2, 4 \pmod{7} \text{ and so } p = C^2 + 7D^2, \\ 0 \pmod{p} & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases} \end{aligned}$$

Proof. Putting  $t = -63$  in Theorem 2.1(i) we have

$$\begin{aligned} &P_{[\frac{p}{4}]} \left( -\frac{65}{63} \right) \\ &\equiv \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k} \binom{2k}{k}}{63^k} \equiv (-1)^{[\frac{p}{4}]} \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k} \binom{2k}{k}}{(-4032)^k} \\ &\equiv - \left( \frac{-6 \cdot (-63)}{p} \right) \sum_{x=0}^{p-1} \left( \frac{-63x^3 + 180x - 144}{p} \right) = - \left( \frac{-42}{p} \right) \sum_{x=0}^{p-1} \left( \frac{7x^3 - 20x + 16}{p} \right) \\ &= - \left( \frac{-42}{p} \right) \sum_{x=0}^{p-1} \left( \frac{7(2x)^3 - 20 \cdot 2x + 16}{p} \right) = - \left( \frac{-21}{p} \right) \sum_{x=0}^{p-1} \left( \frac{7x^3 - 5x + 2}{p} \right) \pmod{p}. \end{aligned}$$

From [R1,R2] we have

$$\sum_{x=0}^{p-1} \left( \frac{x^3 + 21x^2 + 112x}{p} \right) = \begin{cases} -2C \left( \frac{C}{7} \right) & \text{if } p \equiv 1, 2, 4 \pmod{7} \text{ and so } p = C^2 + 7D^2, \\ 0 & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases}$$

As  $x^3 + 21x^2 + 112x = (x+7)^3 - 35(x+7) - 98$  we have

$$\begin{aligned} \sum_{x=0}^{p-1} \left( \frac{x^3 + 21x^2 + 112x}{p} \right) &= \sum_{x=0}^{p-1} \left( \frac{x^3 - 35x - 98}{p} \right) = \sum_{x=0}^{p-1} \left( \frac{(-7x)^3 - 35(-7x) - 98}{p} \right) \\ &= \left( \frac{-1}{p} \right) \sum_{x=0}^{p-1} \left( \frac{7x^3 - 5x + 2}{p} \right) \end{aligned}$$

and so

$$(2.4) \quad \sum_{x=0}^{p-1} \left( \frac{7x^3 - 5x + 2}{p} \right) = \begin{cases} (-1)^{\frac{p+1}{2}} 2C \left( \frac{C}{7} \right) & \text{if } p \equiv 1, 2, 4 \pmod{7} \text{ and } p = C^2 + 7D^2, \\ 0 & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases}$$

Now combining all the above we deduce the result.



**Theorem 2.5.** *Let  $p \neq 2, 3, 7$  be a prime. Then*

$$P_{\frac{p-1}{2}}(3\sqrt{7}/8) \equiv \begin{cases} 2C \pmod{p} & \text{if } p \equiv 1, 9, 25 \pmod{28}, p = C^2 + 7D^2 \text{ and } 4 \mid C - 1, \\ -2\sqrt{7}D \pmod{p} & \text{if } p \equiv 11, 15, 23 \pmod{28}, p = C^2 + 7D^2 \text{ and } 4 \mid D - 1, \\ 0 \pmod{p} & \text{if } p \equiv 3, 5, 6 \pmod{7} \end{cases}$$

and

$$(-1)^{[\frac{p}{4}]} P_{\frac{p-1}{2}}(\sqrt{-63}) \equiv \begin{cases} 2C \pmod{p} & \text{if } p \equiv 1, 9, 25 \pmod{28}, p = C^2 + 7D^2 \text{ and } 4 \mid C - 1, \\ 2D\sqrt{-7} \pmod{p} & \text{if } p \equiv 11, 15, 23 \pmod{28}, p = C^2 + 7D^2 \text{ and } 4 \mid D - 1, \\ 0 \pmod{p} & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases}$$

Proof. From Lemma 2.5 and Theorem 2.4 we have

$$\begin{aligned} & \left(\frac{2 \cdot 64}{p}\right) \left(\frac{3\sqrt{7}}{8}\right)^{\frac{p-1}{2}} P_{\frac{p-1}{2}}\left(\frac{3\sqrt{7}}{8}\right) \\ & \equiv \left(\frac{-63}{p}\right) P_{[\frac{p}{4}]} \left(-\frac{65}{63}\right) \\ & \equiv \begin{cases} 2C(\frac{p}{3})(\frac{C}{7}) \pmod{p} & \text{if } p \equiv 1, 2, 4 \pmod{7} \text{ and so } p = C^2 + 7D^2, \\ 0 \pmod{p} & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases} \end{aligned}$$

Now suppose  $p \equiv 1, 2, 4 \pmod{7}$  and so  $p = C^2 + 7D^2$ . By [S2, p.1317] we have

$$(2.5) \quad 7^{[\frac{p}{4}]} \equiv \begin{cases} (\frac{C}{7}) \pmod{p} & \text{if } p \equiv 1, 9, 25 \pmod{28} \text{ and } C \equiv 1 \pmod{4}, \\ -(\frac{C}{7})\frac{D}{C} \pmod{p} & \text{if } p \equiv 1, 9, 25 \pmod{28} \text{ and } D \equiv 1 \pmod{4}. \end{cases}$$

Thus, noting that  $3^{\frac{p-1}{2}}(\frac{p}{3}) \equiv (\frac{3}{p})(\frac{p}{3}) = (-1)^{\frac{p-1}{2}} \pmod{p}$  we get

$$\begin{aligned} & P_{\frac{p-1}{2}}(3\sqrt{7}/8) \\ & \equiv (3\sqrt{7})^{-\frac{p-1}{2}} 2C\left(\frac{p}{3}\right)\left(\frac{C}{7}\right) \\ & \equiv \begin{cases} 7^{-\frac{p-1}{4}} 2C\left(\frac{C}{7}\right) \equiv 2C \pmod{p} & \text{if } p \equiv 1 \pmod{4} \text{ and } 4 \mid C - 1, \\ -\frac{\sqrt{7}}{7 \cdot 7^{\frac{p-3}{4}}} 2C\left(\frac{C}{7}\right) \equiv \frac{2\sqrt{7}C^2}{7D} \equiv -2\sqrt{7}D \pmod{p} & \text{if } p \equiv 3 \pmod{4} \text{ and } 4 \mid D - 1. \end{cases} \end{aligned}$$

Taking  $x = -63$  in Lemma 2.5 we obtain  $(\sqrt{-7})^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{-63}) \equiv (\sqrt{7})^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\frac{3\sqrt{7}}{8}) \pmod{p}$ . Now combining all the above we obtain the result.

**Theorem 2.6.** *Let  $p$  be an odd prime. Then*

$$P_{\frac{p-1}{2}}\left(\frac{3\sqrt{2}}{4}\right) \equiv \begin{cases} (-1)^{\frac{b}{2}} 2a \pmod{p} & \text{if } 8 \mid p - 1, p = a^2 + 4b^2 \text{ and } 4 \mid a - 1, \\ 4b \pmod{p} & \text{if } 8 \mid p - 5, p = a^2 + 4b^2 \text{ and } 4 \mid b - 1, \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Proof. Taking  $x = 9$  in Lemma 2.5 we have  $\left(\frac{2 \cdot (-8)}{p}\right) \left(\frac{3\sqrt{2}}{4}\right)^{\frac{p-1}{2}} P_{\frac{p-1}{2}}\left(\frac{3\sqrt{2}}{4}\right) \equiv \left(\frac{9}{p}\right) P_{[\frac{p}{4}]}(-\frac{7}{9}) \pmod{p}$ . Thus applying Theorem 2.2 we have

$$\begin{aligned} P_{\frac{p-1}{2}}\left(\frac{3\sqrt{2}}{4}\right) &\equiv \left(\frac{-6}{p}\right) (\sqrt{2})^{\frac{p-1}{2}} P_{[\frac{p}{4}]} \left(-\frac{7}{9}\right) \\ &\equiv \begin{cases} 2^{\frac{p-1}{4}} \cdot 2a \pmod{p} & \text{if } p \equiv 1 \pmod{4}, p = a^2 + 4b^2 \text{ and } 4 \mid a - 1, \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

It is well known that (see [BEW, Theorem 8.2.6])

$$2^{\frac{p-1}{4}} \equiv \begin{cases} (-1)^{\frac{b}{2}} \pmod{p} & \text{if } p \equiv 1 \pmod{8}, \\ \frac{2b}{a} \pmod{p} & \text{if } p \equiv 5 \pmod{8}, p = a^2 + 4b^2 \text{ and } b \equiv a \equiv 1 \pmod{4}. \end{cases}$$

Thus the result follows.

**Theorem 2.7.** *Let  $p$  be an odd prime. Then*

$$P_{\frac{p-1}{2}}(\sqrt{2}) \equiv \begin{cases} (-1)^{\frac{p-1}{8} + \frac{c-1}{4}} 2c \pmod{p} & \text{if } 8 \mid p-1, p = c^2 + 2d^2 \text{ and } c \equiv 1 \pmod{4}, \\ -2\sqrt{2}d \pmod{p} & \text{if } 8 \mid p-3, p = c^2 + 2d^2 \text{ and } d \equiv 1 \pmod{4}, \\ 0 \pmod{p} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Proof. From [BE, Theorems 5.12 and 5.17] we know that

$$\sum_{k=0}^{p-1} \left( \frac{x^3 - 4x^2 + 2x}{p} \right) = \begin{cases} (-1)^{[\frac{p}{8}] + 1} 2c & \text{if } p = c^2 + 2d^2 \equiv 1, 3 \pmod{8} \text{ with } 4 \mid c - 1, \\ 0 & \text{otherwise.} \end{cases}$$

As  $27(x^3 - 4x^2 + 2x) = (3x - 4)^3 - 30(3x - 4) - 56$ , we see that

$$\begin{aligned} &\sum_{x=0}^{p-1} \left( \frac{x^3 - 4x^2 + 2x}{p} \right) \\ &= \left( \frac{3}{p} \right) \sum_{x=0}^{p-1} \left( \frac{(3x - 4)^3 - 30(3x - 4) - 56}{p} \right) = \left( \frac{3}{p} \right) \sum_{x=0}^{p-1} \left( \frac{x^3 - 30x - 56}{p} \right). \end{aligned}$$

Thus, from the above we deduce

$$(2.6) \quad \sum_{x=0}^{p-1} \left( \frac{x^3 - 30x - 56}{p} \right) = \begin{cases} (-1)^{\frac{p+7}{8}} \left(\frac{p}{3}\right) 2c & \text{if } p \equiv 1 \pmod{8}, p = c^2 + 2d^2 \text{ and } 4 \mid c - 1, \\ (-1)^{\frac{p-3}{8}} \left(\frac{p}{3}\right) 2c & \text{if } p \equiv 3 \pmod{8}, p = c^2 + 2d^2 \text{ and } 4 \mid c - 1, \\ 0 & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Taking  $t = 2$  in Theorem 2.1(i) and applying (2.6) we get

$$\begin{aligned}
& \left(\frac{2}{p}\right)(\sqrt{2})^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{2}) \\
& \equiv P_{[\frac{p}{4}]}(0) \equiv \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k} \binom{2k}{k}}{128^k} \equiv -\left(\frac{-12}{p}\right) \sum_{x=0}^{p-1} \left(\frac{2x^3 - 15x - 14}{p}\right) \\
& = -\left(\frac{-3}{p}\right) \sum_{x=0}^{p-1} \left(\frac{2(\frac{x}{2})^3 - 15(\frac{x}{2}) - 14}{p}\right) = -\left(\frac{p}{3}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 30x - 56}{p}\right) \\
& \equiv \begin{cases} (-1)^{\frac{p-1}{8}} 2c \pmod{p} & \text{if } 8 \mid p-1, p = c^2 + 2d^2 \text{ and } 4 \mid c-1, \\ -(-1)^{\frac{p-3}{8}} 2c \pmod{p} & \text{if } 8 \mid p-3, p = c^2 + 2d^2 \text{ and } 4 \mid c-1, \\ 0 \pmod{p} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}
\end{aligned}$$

If  $p \equiv 1 \pmod{8}$  and  $p = c^2 + 2d^2$  with  $c \equiv 1 \pmod{4}$  and  $d = 2^\alpha d_0 (2 \nmid d_0)$ , then  $(\sqrt{2})^{\frac{p-1}{2}} = 2^{\frac{p-1}{4}} \equiv (c/d)^{\frac{p-1}{2}} \equiv (\frac{c}{p})(\frac{d}{p}) = (\frac{p}{c})(\frac{p}{d_0}) = (\frac{c^2+2d^2}{c})(\frac{c^2+2d^2}{d_0}) = (\frac{2}{c}) = (-1)^{\frac{c-1}{4}} \pmod{p}$ .

If  $p \equiv 3 \pmod{8}$  and  $p = c^2 + 2d^2$  with  $c \equiv d \equiv 1 \pmod{4}$ , then  $(\frac{c}{d})^2 \equiv -2 \equiv 2^{\frac{p+1}{2}} \pmod{p}$  and so  $c/d \equiv \pm 2^{\frac{p+1}{4}} \pmod{p}$ . As  $(\frac{c/d}{p}) = (\frac{c}{p})(\frac{d}{p}) = (\frac{c^2+2d^2}{c})(\frac{c^2+2d^2}{d}) = (-1)^{\frac{c^2-1}{8}} = (-1)^{\frac{p-3-2(d^2-1)}{8}} = (-1)^{\frac{p-3}{8}}$ , we see that  $c/d \equiv -(-1)^{\frac{p-3}{8}} 2^{\frac{p+1}{4}} \pmod{p}$ . Hence  $(-1)^{\frac{p-3}{8}} 2c(\sqrt{2})^{-\frac{p-1}{2}} \equiv -2^{\frac{p+1}{4}} d \cdot 2(\sqrt{2})^{1-\frac{p+1}{2}} = -2\sqrt{2}d \pmod{p}$ . Now combining all of the above we obtain the result.

**Remark 2.1.** In [Sul], using Pell sequence Zhi-Wei Sun made a conjecture related to Theorem 2.7.

**Theorem 2.8.** *Let  $p > 3$  be a prime. Then*

$$P_{\frac{p-1}{2}}(\sqrt{-3}) \equiv \begin{cases} (-1)^{\frac{p-1}{4}} 2A \pmod{p} & \text{if } 12 \mid p-1, p = A^2 + 3B^2 \text{ and } 4 \mid A-1, \\ (-1)^{\frac{p-3}{4}} 2B\sqrt{-3} \pmod{p} & \text{if } 12 \mid p-7, p = A^2 + 3B^2 \text{ and } 4 \mid B-1, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3} \end{cases}$$

and

$$P_{\frac{p-1}{2}}\left(\frac{\sqrt{3}}{2}\right) \equiv \begin{cases} 2A \pmod{p} & \text{if } 12 \mid p-1, p = A^2 + 3B^2 \text{ and } 4 \mid A-1, \\ -2\sqrt{3}B \pmod{p} & \text{if } 12 \mid p-7, p = A^2 + 3B^2 \text{ and } 4 \mid B-1, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Proof. From Theorems 2.1 and 2.3 we have

$$\begin{aligned}
& (\sqrt{-3})^{-\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{-3}) \\
& \equiv P_{[\frac{p}{4}]} \left(-\frac{5}{3}\right) \equiv \begin{cases} 2A(\frac{A}{3}) \pmod{p} & \text{if } 3 \mid p-1 \text{ and } p = A^2 + 3B^2, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases}
\end{aligned}$$

Thus the result is true for  $p \equiv 2 \pmod{3}$ . Now suppose  $p \equiv 1 \pmod{3}$ ,  $p = A^2 + 3B^2$  and  $A \equiv 1 \pmod{4}$ . By [S2, p.1317] we have

$$(2.7) \quad 3^{[\frac{p}{4}]} \equiv \begin{cases} (\frac{A}{3}) \pmod{p} & \text{if } p \equiv 1 \pmod{12}, \\ (\frac{A}{3}) \frac{B}{A} \pmod{p} & \text{if } p \equiv 7 \pmod{12} \text{ and } B \equiv 1 \pmod{4}. \end{cases}$$

Thus,

$$P_{\frac{p-1}{2}}(\sqrt{-3}) \equiv \begin{cases} (-3)^{\frac{p-1}{4}} \cdot 2A(\frac{A}{3}) \equiv (-1)^{\frac{p-1}{4}} 2A \pmod{p} & \text{if } p \equiv 1 \pmod{12}, \\ (-3)^{\frac{p-3}{4}} \sqrt{-3} \cdot 2A(\frac{A}{3}) \equiv (-1)^{\frac{p-3}{4}} 2B\sqrt{-3} \pmod{p} & \text{if } 12 \mid p-7 \text{ and } 4 \mid B-1. \end{cases}$$

Taking  $x = -3$  in Lemma 2.5 we obtain  $(\sqrt{3})^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\frac{\sqrt{3}}{2}) \equiv (\sqrt{-3})^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{-3}) \pmod{p}$ . Now combining all the above we obtain the result.

**3. Congruences for  $\sum_{k=0}^{\frac{p-1}{2}} \frac{1}{m^k} \binom{2k}{k}^3 \pmod{p^2}$ .**

**Lemma 3.1.** *Let  $m$  and  $n$  be nonnegative integers. Then*

$$\sum_{k=0}^m \binom{2k}{k} \binom{n+k}{2k} \binom{2k}{k} \binom{k}{m-k} = \sum_{k=0}^m \binom{2k}{k} \binom{n+k}{2k} \binom{2m-2k}{m-k} \binom{n+m-k}{2m-2k}.$$

Lemma 3.1 can be easily proved by using WZ method. For the WZ method one may consult [PWZ]. Clearly the lemma is true for  $m = 0, 1$ . Using Mathematica we find both sides satisfy the same recurrence relation

$$(m+1)(m+2n+2)(m-2n)f(m) + (2m+3)(m^2-2n^2+3m-2n+2)f(m+1) + (m+2)^3 f(m+2) = 0.$$

Thus the lemma is true. The proof certificate for the left hand is

$$R_1(m, k) = -\frac{k^2(2k-m-1)(2k-m)(m+2)}{(m-k+1)(m+2-k)},$$

and the proof certificate for the right hand is

$$R_2(m, k) = \frac{k^2(m-n-k)(m+n-k+1)P(m, k)}{(m-k+1)^2(m-k+2)^2},$$

where

$$P(m, k) = 3mn^2 - 2n^2k + m^2 + 3mn - mk + 6n^2 - 2nk + 4m + 6n - 2k + 4.$$

**Definition 3.1.** *Note that  $\binom{n}{k} \binom{n+k}{k} = \binom{2k}{k} \binom{n+k}{2k}$ . For any nonnegative integer  $n$  we define*

$$S_n(x) = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \binom{2k}{k} x^k = \sum_{k=0}^n \binom{2k}{k}^2 \binom{n+k}{2k} x^k.$$

**Lemma 3.2.** *Let  $n$  be a nonnegative integer. Then*

$$S_n(x) = P_n(\sqrt{1+4x})^2 \quad \text{and} \quad P_n(x)^2 = S_n\left(\frac{x^2-1}{4}\right).$$

Proof. From (2.1) we have

$$\begin{aligned} P_n(x)^2 &= \left( \sum_{k=0}^n \binom{2k}{k} \binom{n+k}{2k} \left(\frac{x-1}{2}\right)^k \right) \left( \sum_{r=0}^n \binom{2r}{r} \binom{n+r}{2r} \left(\frac{x-1}{2}\right)^r \right) \\ &= \sum_{m=0}^{2n} \left(\frac{x-1}{2}\right)^m \sum_{k=0}^m \binom{2k}{k} \binom{n+k}{2k} \binom{2m-2k}{m-k} \binom{n+m-k}{2m-2k}. \end{aligned}$$

On the other hand,

$$\begin{aligned} S_n\left(\frac{x^2-1}{4}\right) &= \sum_{k=0}^n \binom{2k}{k}^2 \binom{n+k}{2k} \left(\frac{x-1}{2}\right)^k \left(1 + \frac{x-1}{2}\right)^k \\ &= \sum_{k=0}^n \binom{2k}{k}^2 \binom{n+k}{2k} \left(\frac{x-1}{2}\right)^k \sum_{r=0}^k \binom{k}{r} \left(\frac{x-1}{2}\right)^r \\ &= \sum_{m=0}^{2n} \left(\frac{x-1}{2}\right)^m \sum_{k=0}^m \binom{2k}{k}^2 \binom{n+k}{2k} \binom{k}{m-k}. \end{aligned}$$

Hence, from the above and Lemma 3.1 we deduce

$$P_n(x)^2 = S_n\left(\frac{x^2-1}{4}\right) \quad \text{and so} \quad P_n(\sqrt{1+4x})^2 = S_n(x).$$

This proves the lemma.

**Theorem 3.1.** *Let  $p$  be an odd prime and  $m \in \mathbb{Z}_p$  with  $m \not\equiv 0 \pmod{p}$ . Then*

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{m^k} \equiv P_{\frac{p-1}{2}} \left( \sqrt{1 - \frac{64}{m}} \right)^2 \pmod{p^2}.$$

Proof. By Definition 3.1 and (1.1) we have

$$S_{\frac{p-1}{2}}\left(-\frac{16}{m}\right) = \sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^2 \binom{\frac{p-1}{2}+k}{2k} \left(-\frac{16}{m}\right)^k \equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{m^k} \pmod{p^2}.$$

On the other hand, by Lemma 3.2 we get

$$S_{\frac{p-1}{2}}\left(-\frac{16}{m}\right) = P_{\frac{p-1}{2}}\left(\sqrt{1+4\left(-\frac{16}{m}\right)}\right)^2 = P_{\frac{p-1}{2}}\left(\sqrt{\frac{m-64}{m}}\right)^2.$$

Thus the result follows.

**Theorem 3.2.** *Let  $p \neq 2, 7$  be a prime. Then*

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^3 \equiv (-1)^{\frac{p-1}{2}} \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{4096^k} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod{7}, \\ 4C^2 \pmod{p} & \text{if } p = C^2 + 7D^2 \equiv 1, 2, 4 \pmod{7}. \end{cases}$$

Proof. Taking  $m = 1, 4096$  in Theorem 3.1 and then applying Theorem 2.5 we deduce the result.

**Remark 3.1** The congruence modulo  $p$  can be deduced from (1.2) and [O, Corollary 11].

**Theorem 3.3.** *Let  $p$  be an odd prime. Then*

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{(-8)^k} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \\ 4a^2 - 2p \pmod{p^2} & \text{if } p = a^2 + 4b^2 \equiv 1 \pmod{4}. \end{cases}$$

Proof. From Theorem 3.1 we have  $\sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^3 / (-8)^k \equiv P_{\frac{p-1}{2}}(3)^2 \pmod{p^2}$ . According to [S3, Corollary 2.3 and Theorem 2.9], we have

$$P_{\frac{p-1}{2}}(3) \equiv \begin{cases} 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}, \\ (-1)^{\frac{p-1}{4}} (2a - \frac{p}{2a}) \pmod{p^2} & \text{if } p = a^2 + 4b^2 \equiv 1 \pmod{4}. \end{cases}$$

Now combining all the above we deduce the result.

**Theorem 3.4.** *Let  $p$  be an odd prime. Then*

$$\begin{aligned} \text{(i)} \quad & \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{(-512)^k} \equiv 0 \pmod{p^2} \quad \text{for } p \equiv 3 \pmod{4}, \\ \text{(ii)} \quad & \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{(-64)^k} \equiv 0 \pmod{p^2} \quad \text{for } p \equiv 5, 7 \pmod{8}, \\ \text{(iii)} \quad & \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{16^k} \equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{256^k} \equiv 0 \pmod{p^2} \quad \text{for } p \equiv 2 \pmod{3}. \end{aligned}$$

Proof. Putting  $m = -512, -64, 16, 256$  in Theorem 3.1 and then applying Theorems 2.6-2.8 we deduce the result.

**Remark 3.2** Theorems 3.3 and 3.4 were conjectured by Zhi-Wei Sun in [Su1, Su3].

#### 4. A general congruence modulo $p^2$ .

**Lemma 4.1.** *Let  $m$  be a nonnegative integer. Then*

$$\sum_{k=0}^m \binom{2k}{k}^2 \binom{4k}{2k} \binom{k}{m-k} (-64)^{m-k} = \sum_{k=0}^m \binom{2k}{k} \binom{4k}{2k} \binom{2(m-k)}{m-k} \binom{4(m-k)}{2(m-k)}.$$

We prove the lemma by using WZ method and Mathematica. Clearly the result is true for  $m = 0, 1$ . Since both sides satisfy the same recurrence relation

$$1024(m+1)(2m+1)(2m+3)S(m) - 8(2m+3)(8m^2+24m+19)S(m+1) + (m+2)^3S(m+2) = 0,$$

we see that Lemma 4.1 is true. The proof certificate for the left hand side is

$$-\frac{4096k^2(m+2)(m-2k)(m-2k+1)}{(m-k+1)(m-k+2)},$$

and the proof certificate for the right hand side is

$$\frac{16k^2(4m-4k+1)(4m-4k+3)(16m^2-16mk+55m-26k+46)}{(m-k+1)^2(m-k+2)^2}.$$

**Theorem 4.1.** *Let  $p$  be an odd prime and let  $x$  be a variable. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} (x(1-64x))^k \equiv \left( \sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \right)^2 \pmod{p^2}.$$

Proof. It is clear that

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} (x(1-64x))^k \\ &= \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} x^k \sum_{r=0}^k \binom{k}{r} (-64x)^r \\ &= \sum_{m=0}^{2(p-1)} x^m \sum_{k=0}^{\min\{m, p-1\}} \binom{2k}{k}^2 \binom{4k}{2k} \binom{k}{m-k} (-64)^{m-k}. \end{aligned}$$

Suppose  $p \leq m \leq 2p-2$  and  $0 \leq k \leq p-1$ . If  $k > \frac{p}{2}$ , then  $p \mid \binom{2k}{k}$  and so  $p^2 \mid \binom{2k}{k}^2$ . If  $k < \frac{p}{2}$ , then  $m-k \geq p-k > k$  and so  $\binom{k}{m-k} = 0$ . Thus, from the above and Lemma 4.1 we deduce

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} (x(1-64x))^k \\ &\equiv \sum_{m=0}^{p-1} x^m \sum_{k=0}^m \binom{2k}{k}^2 \binom{4k}{2k} \binom{k}{m-k} (-64)^{m-k} \\ &= \sum_{m=0}^{p-1} x^m \sum_{k=0}^m \binom{2k}{k} \binom{4k}{2k} \binom{2(m-k)}{m-k} \binom{4(m-k)}{2(m-k)} \\ &= \sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \sum_{m=k}^{p-1} \binom{2(m-k)}{m-k} \binom{4(m-k)}{2(m-k)} x^{m-k} \\ &= \sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \sum_{r=0}^{p-1-k} \binom{2r}{r} \binom{4r}{2r} x^r \\ &= \left( \sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \right)^2 - \sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \sum_{r=p-k}^{p-1} \binom{2r}{r} \binom{4r}{2r} x^r \pmod{p^2}. \end{aligned}$$

Now suppose  $0 \leq k \leq p-1$  and  $p-k \leq r \leq p-1$ . If  $k \geq \frac{3p}{4}$ , then  $p^2 \nmid (2k)!$ ,  $p^3 \mid (4k)!$  and so  $\binom{2k}{k} \binom{4k}{2k} = \frac{(4k)!}{(2k)!k!^2} \equiv 0 \pmod{p^2}$ . If  $k < \frac{p}{4}$ , then  $r \geq p-k > \frac{3p}{4}$  and so  $\binom{2r}{r} \binom{4r}{2r} = \frac{(4r)!}{(2r)!r!^2} \equiv 0 \pmod{p^2}$ . If  $\frac{p}{4} < k < \frac{p}{2}$ , then  $r \geq p-k > \frac{p}{2}$ ,  $p \mid \binom{2r}{r}$  and  $p \mid \binom{4k}{2k}$ . If  $\frac{p}{2} < k < \frac{3p}{4}$ , then  $r \geq p-k > \frac{p}{4}$ ,  $p \mid \binom{2k}{k}$  and  $p \mid \binom{2r}{r} \binom{4r}{2r}$ . Hence we always have  $\binom{2k}{k} \binom{4k}{2k} \binom{2r}{r} \binom{4r}{2r} \equiv 0 \pmod{p^2}$  and so

$$\sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \sum_{r=p-k}^{p-1} \binom{2r}{r} \binom{4r}{2r} x^r \equiv 0 \pmod{p^2}.$$

Now combining all the above we obtain the result.

**Corollary 4.1.** *Let  $p$  be an odd prime and  $m \in \mathbb{Z}_p$  with  $m \not\equiv 0 \pmod{p}$ . Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \equiv \left( \sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} \left( \frac{1 - \sqrt{1 - 256/m}}{128} \right)^k \right)^2 \pmod{p^2}.$$

Proof. Taking  $x = \frac{1 - \sqrt{1 - 256/m}}{128}$  in Theorem 4.1 we deduce the result.

**Lemma 4.2.** *Let  $p$  be a prime greater than 3. Then*

$$P_{[\frac{p}{4}]}(\sqrt{t}) \equiv - \sum_{x=0}^{p-1} (x^3 + 4x^2 + 2(1 - \sqrt{t})x)^{\frac{p-1}{2}} \pmod{p}.$$

Proof. It is clear that

$$\begin{aligned} & \sum_{x=0}^{p-1} (x^3 + 4x^2 + 2(1 - \sqrt{t})x)^{\frac{p-1}{2}} \\ & \equiv \sum_{x=0}^{p-1} \left( \left( x - \frac{4}{3} \right)^3 + 4 \left( x - \frac{4}{3} \right)^2 + 2(1 - \sqrt{t}) \left( x - \frac{4}{3} \right) \right)^{\frac{p-1}{2}} \\ & = \sum_{x=0}^{p-1} \left( x^3 - \frac{2(3\sqrt{t} + 5)}{3}x + \frac{8(9\sqrt{t} + 7)}{27} \right)^{\frac{p-1}{2}} \\ & \equiv \sum_{x=0}^{p-1} \left( \left( \frac{2x}{3} \right)^3 - \frac{2(3\sqrt{t} + 5)}{3} \cdot \frac{2x}{3} + \frac{8(9\sqrt{t} + 7)}{27} \right)^{\frac{p-1}{2}} \\ & = \left( \frac{8}{27} \right)^{\frac{p-1}{2}} \sum_{x=0}^{p-1} \left( x^3 - \frac{3}{2}(3\sqrt{t} + 5)x + 9\sqrt{t} + 7 \right)^{\frac{p-1}{2}} \pmod{p}. \end{aligned}$$

By Theorem 2.1(ii),

$$(4.1) \quad P_{[\frac{p}{4}]}(u) \equiv - \left( \frac{6}{p} \right) \sum_{x=0}^{p-1} \left( x^3 - \frac{3}{2}(3u + 5)x + 9u + 7 \right)^{\frac{p-1}{2}} \pmod{p}$$

for  $u = 0, 1, \dots, p-1$ . Since both sides are polynomials of  $u$  with degree at most  $(p-1)/2$ . By Lagrange's theorem, (4.1) is also true when  $u$  is a variable. Hence

$$P_{[\frac{p}{4}]}(\sqrt{t}) \equiv - \left( \frac{6}{p} \right) \sum_{x=0}^{p-1} \left( x^3 - \frac{3}{2}(3\sqrt{t} + 5)x + 9\sqrt{t} + 7 \right)^{\frac{p-1}{2}} \pmod{p}.$$

Now combining all the above with the fact  $\left( \frac{8}{27} \right)^{\frac{p-1}{2}} \equiv \left( \frac{6}{p} \right) \pmod{p}$  we deduce the result.



**Theorem 4.2.** Let  $p$  be an odd prime,  $m \in \mathbb{Z}_p$ ,  $m \not\equiv 0 \pmod{p}$  and  $t = \sqrt{1 - 256/m}$ . Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \equiv P_{[\frac{p}{4}]}(t)^2 \equiv \left( \sum_{x=0}^{p-1} (x^3 + 4x^2 + 2(1-t)x)^{\frac{p-1}{2}} \right)^2 \pmod{p}.$$

Moreover, if  $P_{[\frac{p}{4}]}(t) \equiv 0 \pmod{p}$  or  $\sum_{x=0}^{p-1} (x^3 + 4x^2 + 2(1-t)x)^{\frac{p-1}{2}} \equiv 0 \pmod{p}$ , then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \equiv 0 \pmod{p^2}.$$

Proof. For  $\frac{p}{2} < k < p$ ,  $\binom{2k}{k} = \frac{(2k)!}{k!^2} \equiv 0 \pmod{p}$ . For  $\frac{p}{4} < k < \frac{p}{2}$ ,  $\binom{4k}{2k} = \frac{(4k)!}{(2k)!^2} \equiv 0 \pmod{p}$ . Thus,  $p \mid \binom{2k}{k} \binom{4k}{2k}$  for  $\frac{p}{4} < k < p$ . Hence, by Lemmas 2.2 and 4.2,

$$P_{[\frac{p}{4}]}(t) \equiv \sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} \left( \frac{1-t}{128} \right)^k \equiv - \sum_{x=0}^{p-1} (x^3 + 4x^2 + 2(1-t)x)^{\frac{p-1}{2}} \pmod{p}.$$

This together with Corollary 4.1 gives the result.

**Theorem 4.3.** Let  $p \equiv 1, 3 \pmod{8}$  be a prime and so  $p = c^2 + 2d^2$  with  $c, d \in \mathbb{Z}$  and  $c \equiv 1 \pmod{4}$ . Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \equiv (-1)^{[\frac{p}{8}] + \frac{p-1}{2}} \left( 2c - \frac{p}{2c} \right) \pmod{p^2}.$$

Proof. By the proof of Theorem 2.7,

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \equiv \sum_{k=0}^{[p/4]} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \equiv (-1)^{[\frac{p}{8}] + \frac{p-1}{2}} 2c \pmod{p}.$$

Set  $\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} = (-1)^{[\frac{p}{8}] + \frac{p-1}{2}} 2c + qp$ . Then

$$\left( \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \right)^2 = ((-1)^{[\frac{p}{8}] + \frac{p-1}{2}} 2c + qp)^2 \equiv 4c^2 + (-1)^{[\frac{p}{8}] + \frac{p-1}{2}} 4cqp \pmod{p^2}.$$

Taking  $x = \frac{1}{128}$  in Theorem 4.1 we get

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{256^k} \equiv \left( \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \right)^2 \pmod{p^2}.$$

From [M] and [Su2] we have  $\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} / 256^k \equiv 4c^2 - 2p \pmod{p^2}$ . Thus

$$4c^2 - 2p \equiv \left( \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \right)^2 \equiv 4c^2 + (-1)^{[\frac{p}{8}] + \frac{p-1}{2}} 4cqp \pmod{p^2}$$

and hence  $q \equiv -(-1)^{[\frac{p}{8}] + \frac{p-1}{2}} \frac{1}{2c} \pmod{p}$ . So the theorem is proved.

We note that Theorem 4.3 was conjectured by Zhi-Wei Sun in [Su1].

## 5. Congruences for $\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} / m^k$ .

**Theorem 5.1.** *Let  $p \neq 2, 3, 7$  be a prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{648^k} &\equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \\ 4a^2 \pmod{p} & \text{if } p = a^2 + 4b^2 \equiv 1 \pmod{4}, \end{cases} \\ \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-144)^k} &\equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}, \\ 4A^2 \pmod{p} & \text{if } p = A^2 + 3B^2 \equiv 1 \pmod{3}, \end{cases} \\ \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-3969)^k} &\equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod{7}, \\ 4C^2 \pmod{p} & \text{if } p = C^2 + 7D^2 \equiv 1, 2, 4 \pmod{7}. \end{cases} \end{aligned}$$

Proof. Taking  $m = 648, -144, -3969$  in Theorem 4.2 and then applying Theorems 2.2-2.4 and (1.6) we deduce the result.

We remark that Theorem 5.1 was conjectured by the author in [S3].

**Lemma 5.1 ([S4, Lemma 4.1]).** *Let  $p$  be an odd prime and let  $a, m, n$  be  $p$ -adic integers. Then*

$$\sum_{x=0}^{p-1} (x^3 + a^2mx + a^3n)^{\frac{p-1}{2}} \equiv a^{\frac{p-1}{2}} \sum_{x=0}^{p-1} (x^3 + mx + n)^{\frac{p-1}{2}} \pmod{p}.$$

Moreover, if  $a, m, n$  are congruent to some integers modulo  $p$ , then

$$\sum_{x=0}^{p-1} \left( \frac{x^3 + a^2mx + a^3n}{p} \right) = \left( \frac{a}{p} \right) \sum_{x=0}^{p-1} \left( \frac{x^3 + mx + n}{p} \right).$$

**Theorem 5.2.** *Let  $p \neq 2, 3, 7$  be a prime. Then*

$$\begin{aligned} P_{[\frac{p}{4}]} \left( \frac{5\sqrt{-7}}{9} \right) &\equiv \begin{cases} \left( \frac{3(7+\sqrt{-7})}{p} \right) \left( \frac{C}{7} \right) 2C \pmod{p} & \text{if } p \equiv 1, 2, 4 \pmod{7} \text{ and so } p = C^2 + 7D^2, \\ 0 \pmod{p} & \text{if } p \equiv 3, 5, 6 \pmod{7} \end{cases} \end{aligned}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{81^k} \equiv \begin{cases} 4C^2 \pmod{p} & \text{if } p \equiv 1, 2, 4 \pmod{7} \text{ and so } p = C^2 + 7D^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases}$$

Proof. By (4.1),

$$P_{[\frac{p}{4}]} \left( \frac{5\sqrt{-7}}{9} \right) \equiv - \left( \frac{6}{p} \right) \sum_{x=0}^{p-1} \left( x^3 - \frac{5}{2} (3 + \sqrt{-7})x + 7 + 5\sqrt{-7} \right)^{\frac{p-1}{2}} \pmod{p}.$$

Since

$$\frac{-\frac{5}{2}(3 + \sqrt{-7})}{-35} = \left( \frac{1 - \sqrt{-7}}{2\sqrt{-7}} \right)^2 \quad \text{and} \quad \frac{7 + 5\sqrt{-7}}{-98} = \left( \frac{1 - \sqrt{-7}}{2\sqrt{-7}} \right)^3,$$

by the above and Lemma 5.1 we have

$$P_{[\frac{p}{4}]} \left( \frac{5\sqrt{-7}}{9} \right) \equiv - \left( \frac{6}{p} \right) \left( \frac{1 - \sqrt{-7}}{2\sqrt{-7}} \right)^{\frac{p-1}{2}} \sum_{x=0}^{p-1} \left( \frac{x^3 - 35x - 98}{p} \right) \pmod{p}.$$

By the proof of Theorem 2.4,

$$(5.1) \quad \sum_{x=0}^{p-1} \left( \frac{x^3 - 35x - 98}{p} \right) = \begin{cases} -2C(\frac{C}{7}) & \text{if } p = C^2 + 7D^2 \equiv 1, 2, 4 \pmod{7}, \\ 0 & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases}$$

For  $p \equiv 1, 2, 4 \pmod{7}$  we see that

$$\left( \frac{6}{p} \right) \left( \frac{1 - \sqrt{-7}}{2\sqrt{-7}} \right)^{\frac{p-1}{2}} = \left( \frac{6}{p} \right) \left( \frac{7 + \sqrt{-7}}{2 \cdot (-7)} \right)^{\frac{p-1}{2}} \equiv \left( \frac{3}{p} \right) \left( \frac{7 + \sqrt{-7}}{p} \right) \pmod{p}.$$

Thus, from the above we deduce the congruence for  $P_{[\frac{p}{4}]}(\frac{5\sqrt{-7}}{9}) \pmod{p}$ . Applying Theorem 4.2 (with  $m = 81$ ) we obtain the remaining result.

**Theorem 5.3.** *Let  $p$  be a prime such that  $p \equiv 1, 9 \pmod{20}$  and hence  $p = u^2 + 5v^2$  for some integers  $u$  and  $v$ . Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-1024)^k} \equiv 4u^2 \pmod{p}.$$

Proof. Taking  $m = -1024$  and  $t = \sqrt{5}/2$  in Theorem 4.2 we see that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-1024)^k} \equiv \left( \sum_{x=0}^{p-1} \left( \frac{x^3 + 4x^2 + (2 - \sqrt{5})x}{p} \right) \right)^2 \pmod{p}.$$

By [LM, Theorem 11] we have

$$\sum_{x=0}^{p-1} \left( \frac{x^3 + 4x^2 + (2 - \sqrt{5})x}{p} \right) = \pm 2u.$$

Thus the result follows.

**Remark 5.1** Let  $p \equiv 1 \pmod{20}$  be a prime and hence  $p = a^2 + 4b^2 = u^2 + 5v^2$  with  $a, b, u, v \in \mathbb{Z}$ . A result of Cauchy ([BEW, p. 291]) states that

$$\left( \frac{\frac{p-1}{2}}{20} \right)^2 \equiv \begin{cases} 4u^2 \pmod{p} & \text{if } 5 \nmid a, \\ -4u^2 \pmod{p} & \text{if } 5 \mid a. \end{cases}$$

Let  $m \in \{5, 13, 37\}$  and  $f(m) = -1024, -82944, -2^{10} \cdot 21^4$  according as  $m = 5, 13$  or  $37$ . Suppose that  $p$  is an odd prime such that  $p \nmid mf(m)$ . In [Su1], Z.W. Sun conjectured that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{f(m)^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left( \frac{m}{p} \right) = \left( \frac{-1}{p} \right) = 1 \text{ and so } p = x^2 + my^2, \\ 2p - 2x^2 \pmod{p^2} & \text{if } \left( \frac{m}{p} \right) = \left( \frac{-1}{p} \right) = -1 \text{ and so } 2p = x^2 + my^2, \\ 0 \pmod{p^2} & \text{if } \left( \frac{m}{p} \right) = -\left( \frac{-1}{p} \right). \end{cases}$$

Let  $p > 3$  be a prime and let  $\mathbb{F}_p$  be the field of  $p$  elements. For  $m, n \in \mathbb{F}_p$  let  $\#E_p(x^3 + mx + n)$  be the number of points on the curve  $E_p: y^2 = x^3 + mx + n$  over the field  $\mathbb{F}_p$ . It is well known that

$$(5.2) \quad \#E_p(x^3 + mx + n) = p + 1 + \sum_{x=0}^{p-1} \left( \frac{x^3 + mx + n}{p} \right).$$

Let  $K = \mathbb{Q}(\sqrt{-d})$  be an imaginary quadratic field and the curve  $y^2 = x^3 + mx + n$  has complex multiplication by  $K$ . By Deuring's theorem ([C, Theorem 14.16], [PV], [I]), we have

$$(5.3) \quad \#E_p(x^3 + mx + n) = \begin{cases} p + 1 & \text{if } p \text{ is inert in } K, \\ p + 1 - \pi - \bar{\pi} & \text{if } p = \pi\bar{\pi} \text{ in } K, \end{cases}$$

where  $\pi$  is in an order in  $K$  and  $\bar{\pi}$  is the conjugate number of  $\pi$ . If  $4p = u^2 + dv^2$  with  $u, v \in \mathbb{Z}$ , we may take  $\pi = \frac{1}{2}(u + v\sqrt{-d})$ . Thus,

$$(5.4) \quad \sum_{x=0}^{p-1} \left( \frac{x^3 + mx + n}{p} \right) = \begin{cases} \pm u & \text{if } 4p = u^2 + dv^2 \text{ with } u, v \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

In [JM] and [PV] the sign of  $u$  in (5.4) was determined for those imaginary quadratic fields  $K$  with class number 1. In [LM] and [I] the sign of  $u$  in (5.4) was determined for imaginary quadratic fields  $K$  with class number 2.

**Theorem 5.4.** *Let  $p$  be a prime such that  $p \equiv \pm 1 \pmod{12}$ . Then*

$$P_{[\frac{p}{4}]} \left( \frac{7}{12} \sqrt{3} \right) \equiv \begin{cases} \left( \frac{2+2\sqrt{3}}{p} \right) 2x \pmod{p} & \text{if } p = x^2 + 9y^2 \equiv 1 \pmod{12} \text{ with } 3 \mid x - 1, \\ 0 \pmod{p} & \text{if } p \equiv 11 \pmod{12} \end{cases}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-12288)^k} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p = x^2 + 9y^2 \equiv 1 \pmod{12}, \\ 0 \pmod{p^2} & \text{if } p \equiv 11 \pmod{12}. \end{cases}$$

*Proof.* From [I, p.133] we know that the elliptic curve defined by the equation  $y^2 = x^3 - (120 + 42\sqrt{3})x + 448 + 336\sqrt{3}$  has complex multiplication by the order of discriminant  $-36$ . Thus, by (5.4) and [I, Theorem 3.1] we have

$$\begin{aligned} & \sum_{n=0}^{p-1} \left( \frac{n^3 - (120 + 42\sqrt{3})n + 448 + 336\sqrt{3}}{p} \right) \\ &= \begin{cases} -2x \left( \frac{1+\sqrt{3}}{p} \right) & \text{if } p = x^2 + 9y^2 \equiv 1 \pmod{12} \text{ with } 3 \mid x - 1, \\ 0 & \text{if } p \equiv 11 \pmod{12}. \end{cases} \end{aligned}$$

By (4.1),

$$\begin{aligned} P_{[\frac{p}{4}]} \left( \frac{7}{12} \sqrt{3} \right) &\equiv - \left( \frac{6}{p} \right) \sum_{n=0}^{p-1} \left( n^3 - \frac{60 + 21\sqrt{3}}{8} n + \frac{28 + 21\sqrt{3}}{4} \right)^{\frac{p-1}{2}} \\ &\equiv - \left( \frac{6}{p} \right) \sum_{n=0}^{p-1} \left( \left( \frac{n}{4} \right)^3 - \frac{60 + 21\sqrt{3}}{8} \cdot \frac{n}{4} + \frac{28 + 21\sqrt{3}}{4} \right)^{\frac{p-1}{2}} \\ &\equiv - \left( \frac{6}{p} \right) \sum_{n=0}^{p-1} \left( \frac{n^3 - (120 + 42\sqrt{3})n + 448 + 336\sqrt{3}}{p} \right) \pmod{p}. \end{aligned}$$

Now combining all the above we obtain the congruence for  $P_{[\frac{7}{4}]}(\frac{7}{12}\sqrt{3}) \pmod{p}$ . Applying Theorem 4.2 we deduce the remaining result.

**Remark 5.2** In [Su1, Conjecture A24], Z.W. Sun conjectured that for any prime  $p > 3$ ,

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-12288)^k} \equiv \begin{cases} (-1)^{[\frac{p}{6}]}(4x^2 - 2p) \pmod{p} & \text{if } p = x^2 + y^2 \equiv 1 \pmod{12} \text{ and } 4 \mid x-1, \\ -4(\frac{xy}{3})xy \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv 5 \pmod{12} \text{ and } 4 \mid x-1, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

**Theorem 5.5.** Let  $p$  be an odd prime such that  $p \neq 3$  and  $(\frac{13}{p}) = 1$ . Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-82944)^k} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p = x^2 + 13y^2 \equiv 1 \pmod{4}, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Proof. From [LM, Table II] we know that the elliptic curve defined by the equation  $y^2 = x^3 + 4x^2 + (2 - \frac{5}{9}\sqrt{13})x$  has complex multiplication by the order of discriminant  $-52$ . Thus, by (5.4) we have

$$\sum_{n=0}^{p-1} \left( \frac{n^3 + 4n^2 + (2 - \frac{5}{9}\sqrt{13})n}{p} \right) = \begin{cases} 2x & \text{if } p \equiv 1 \pmod{4} \text{ and so } p = x^2 + 13y^2, \\ 0 & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Now taking  $m = -2^{10} \cdot 3^4$  and  $t = \frac{5}{18}\sqrt{13}$  in Theorem 4.2 and applying the above we deduce the result.

**Theorem 5.6.** Let  $p$  be an odd prime such that  $p \neq 3, 7$  and  $(\frac{37}{p}) = 1$ . Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-2^{10} \cdot 21^4)^k} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p \equiv 1 \pmod{4} \text{ and so } p = x^2 + 37y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Proof. From [LM, Table II] we know that the elliptic curve defined by the equation  $y^2 = x^3 + 4x^2 + (2 - \frac{145}{441}\sqrt{37})x$  has complex multiplication by the order of discriminant  $-148$ . Thus, by (5.4) we have

$$\sum_{n=0}^{p-1} \left( \frac{n^3 + 4n^2 + (2 - \frac{145}{441}\sqrt{37})n}{p} \right) = \begin{cases} 2x & \text{if } p \equiv 1 \pmod{4} \text{ and so } p = x^2 + 37y^2, \\ 0 & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Now taking  $m = -2^{10} \cdot 21^4$  and  $t = \frac{145}{882}\sqrt{37}$  in Theorem 4.2 and applying the above we deduce the result.

Let  $b \in \{3, 5, 11, 29\}$  and  $f(b) = 48^2, 12^4, 1584^2, 396^4$  according as  $b = 3, 5, 11, 29$ . For any odd prime  $p$  with  $p \nmid bf(b)$ , Z.W. Sun conjectured that ([Su1, Conjectures A14, A16, A18 and A21])

$$(5.5) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{f(b)^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{2}{p}) = (\frac{-b}{p}) = 1 \text{ and so } p = x^2 + 2by^2, \\ 2p - 8x^2 \pmod{p^2} & \text{if } (\frac{2}{p}) = (\frac{-b}{p}) = -1 \text{ and so } p = 2x^2 + by^2, \\ 0 \pmod{p^2} & \text{if } (\frac{2}{p}) = -(\frac{-b}{p}). \end{cases}$$

Now we partially solve the above conjecture.

**Theorem 5.7.** *Let  $p$  be an odd prime such that  $p \equiv \pm 1 \pmod{8}$ . Then*

$$P_{[\frac{p}{4}]} \left( \frac{2\sqrt{2}}{3} \right) \equiv \begin{cases} (-1)^{\frac{p-1}{2}} \left( \frac{\sqrt{2}}{p} \right) \left( \frac{x}{3} \right) 2x \pmod{p} & \text{if } p = x^2 + 6y^2 \equiv 1, 7 \pmod{24}, \\ 0 \pmod{p} & \text{if } p \equiv 17, 23 \pmod{24} \end{cases}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{48^{2k}} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p = x^2 + 6y^2 \equiv 1, 7 \pmod{24}, \\ 0 \pmod{p^2} & \text{if } p \equiv 17, 23 \pmod{24}. \end{cases}$$

Proof. From [I, p.133] we know that the elliptic curve defined by the equation  $y^2 = x^3 + (-21 + 12\sqrt{2})x - 28 + 22\sqrt{2}$  has complex multiplication by the order of discriminant  $-24$ . Thus, by (5.4) and [I, Theorem 3.1] we have

$$\begin{aligned} & \sum_{n=0}^{p-1} \left( \frac{n^3 + (-21 + 12\sqrt{2})n - 28 + 22\sqrt{2}}{p} \right) \\ &= \begin{cases} 2x \left( \frac{2x}{3} \right) \left( \frac{1+\sqrt{2}}{p} \right) & \text{if } p \equiv 1, 7 \pmod{24} \text{ and so } p = x^2 + 6y^2, \\ 0 & \text{if } p \equiv 17, 23 \pmod{24}. \end{cases} \end{aligned}$$

By (4.1),

$$P_{[\frac{p}{4}]} \left( \frac{2\sqrt{2}}{3} \right) \equiv - \left( \frac{6}{p} \right) \sum_{n=0}^{p-1} \left( n^3 - \frac{15 + 6\sqrt{2}}{2}n + 7 + 6\sqrt{2} \right)^{\frac{p-1}{2}} \pmod{p}.$$

Since

$$\frac{-(15 + 6\sqrt{2})/2}{-21 + 12\sqrt{2}} = \left( \frac{\sqrt{2} + 1}{\sqrt{2}} \right)^2 \quad \text{and} \quad \frac{7 + 6\sqrt{2}}{-28 + 22\sqrt{2}} = \left( \frac{\sqrt{2} + 1}{\sqrt{2}} \right)^3,$$

by Lemma 5.1 and the above we have

$$\begin{aligned} P_{[\frac{p}{4}]} \left( \frac{2\sqrt{2}}{3} \right) &\equiv - \left( \frac{6}{p} \right) \left( \frac{\sqrt{2}(\sqrt{2} + 1)}{p} \right) \sum_{n=0}^{p-1} \left( \frac{n^3 + (-21 + 12\sqrt{2})n - 28 + 22\sqrt{2}}{p} \right) \\ &\equiv \begin{cases} - \left( \frac{6}{p} \right) \left( \frac{\sqrt{2}}{p} \right) 2x \left( \frac{2x}{3} \right) \pmod{p} & \text{if } p = x^2 + 6y^2 \equiv 1, 7 \pmod{24}, \\ 0 \pmod{p} & \text{if } p \equiv 17, 23 \pmod{24}. \end{cases} \end{aligned}$$

This yields the result for  $P_{[\frac{p}{4}]} \left( \frac{2\sqrt{2}}{3} \right) \pmod{p}$ . Taking  $m = 48^2$  and  $t = \frac{2}{3}\sqrt{2}$  in Theorem 4.2 and applying the above we deduce the remaining result.

**Theorem 5.8.** *Let  $p$  be a prime such that  $p \equiv \pm 1 \pmod{5}$ . Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{12^{4k}} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p = x^2 + 10y^2 \equiv 1, 9, 11, 19 \pmod{40}, \\ 0 \pmod{p^2} & \text{if } p \equiv 21, 29, 31, 39 \pmod{40}. \end{cases}$$

Proof. From [LM, Table II] we know that the elliptic curve defined by the equation  $y^2 = x^3 + 4x^2 + (2 - \frac{8}{9}\sqrt{5})x$  has complex multiplication by the order of discriminant  $-40$ . Thus, by (5.4) we have

$$\sum_{n=0}^{p-1} \left( \frac{n^3 + 4n^2 + (2 - \frac{8}{9}\sqrt{5})n}{p} \right) = \begin{cases} 2x & \text{if } p \equiv 1, 9, 11, 19 \pmod{40} \text{ and so } p = x^2 + 10y^2, \\ 0 & \text{if } p \equiv 21, 29, 31, 39 \pmod{40}. \end{cases}$$

Now taking  $m = 12^4$  and  $t = \frac{4}{9}\sqrt{5}$  in Theorem 4.2 and applying the above we deduce the result.

**Theorem 5.9.** *Let  $p$  be a prime such that  $p \equiv \pm 1 \pmod{8}$ . Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{1584^{2k}} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } \left(\frac{p}{11}\right) = 1 \text{ and so } p = x^2 + 22y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{11}\right) = -1. \end{cases}$$

Proof. From [LM, Table II] we know that the elliptic curve defined by the equation  $y^2 = x^3 + 4x^2 + (2 - \frac{140}{99}\sqrt{2})x$  has complex multiplication by the order of discriminant  $-88$ . Thus, by (5.4) we have

$$\sum_{n=0}^{p-1} \left( \frac{n^3 + 4n^2 + (2 - \frac{140}{99}\sqrt{2})n}{p} \right) = \begin{cases} 2x & \text{if } \left(\frac{p}{11}\right) = 1 \text{ and so } p = x^2 + 22y^2, \\ 0 & \text{if } \left(\frac{p}{11}\right) = -1. \end{cases}$$

Now taking  $m = 1584^2$  and  $t = \frac{70}{99}\sqrt{2}$  in Theorem 4.2 and applying the above we deduce the result.

**Theorem 5.10.** *Let  $p$  be an odd prime such that  $\left(\frac{29}{p}\right) = 1$ . Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{396^{4k}} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p = x^2 + 58y^2 \equiv 1, 3 \pmod{8}, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Proof. From [LM, Table II] we know that the elliptic curve defined by the equation  $y^2 = x^3 + 4x^2 + (2 - \frac{3640}{9801}\sqrt{29})x$  has complex multiplication by the order of discriminant  $-232$ . Thus, by (5.4) we have

$$\sum_{n=0}^{p-1} \left( \frac{n^3 + 4n^2 + (2 - \frac{3640}{9801}\sqrt{29})n}{p} \right) = \begin{cases} 2x & \text{if } p \equiv 1, 3 \pmod{8} \text{ and so } p = x^2 + 58y^2, \\ 0 & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Now taking  $m = 396^4$  and  $t = \frac{1820}{9801}\sqrt{29}$  in Theorem 4.2 and applying the above we deduce the result.

**Theorem 5.11.** *Let  $p$  be an odd prime such that  $p \equiv 1, 5, 19, 23 \pmod{24}$ . Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{28^{4k}} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p \equiv 1, 19 \pmod{24} \text{ and so } p = x^2 + 18y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 23 \pmod{24}. \end{cases}$$

Proof. From [LM, Table II] we know that the elliptic curve defined by the equation  $y^2 = x^3 + 4x^2 + (2 - \frac{40}{49}\sqrt{6})x$  has complex multiplication by the order of discriminant  $-72$ . Thus, by (5.4) we have

$$\sum_{n=0}^{p-1} \left( \frac{n^3 + 4n^2 + (2 - \frac{40}{49}\sqrt{6})n}{p} \right) = \begin{cases} 2x & \text{if } p \equiv 1, 19 \pmod{24} \text{ and so } p = x^2 + 18y^2, \\ 0 & \text{if } p \equiv 5, 23 \pmod{24}. \end{cases}$$

Now taking  $m = 28^4$  and  $t = \frac{20}{49}\sqrt{6}$  in Theorem 4.2 and applying the above we deduce the result.

**Remark 5.3** Let  $p \neq 2, 7$  be a prime. Z.W. Sun conjectured that ([Su1, Conjecture A28])

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{28^{4k}} \equiv \begin{cases} 4x^2 - 2p \pmod{p} & \text{if } p = x^2 + 2y^2 \equiv 1, 3 \pmod{8}, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

**Theorem 5.12.** *Let  $p$  be an odd prime such that  $p \equiv \pm 1 \pmod{5}$ . Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-2^{14} \cdot 3^4 \cdot 5)^k} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p = x^2 + 25y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Proof. From [LM, Table II] we know that the elliptic curve defined by the equation  $y^2 = x^3 + 4x^2 + (2 - \frac{161}{180}\sqrt{5})x$  has complex multiplication by the order of discriminant  $-100$ . Thus, by (5.4) we have

$$\sum_{n=0}^{p-1} \left( \frac{n^3 + 4n^2 + (2 - \frac{161}{180}\sqrt{5})n}{p} \right) = \begin{cases} 2x & \text{if } p = x^2 + 25y^2, \\ 0 & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Now taking  $m = -2^{14} \cdot 3^4 \cdot 5$  and  $t = \frac{161}{360}\sqrt{5}$  in Theorem 4.2 and applying the above we deduce the result.

**Remark 5.4** Let  $p > 5$  be a prime. Z.W. Sun made a conjecture ([Su1, Conjecture A25]) equivalent to

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-2^{14} \cdot 3^4 \cdot 5)^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p} & \text{if } p = x^2 + 25y^2, \\ -4xy \pmod{p^2} & \text{if } p = x^2 + y^2 \text{ with } 5 \mid x - y, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

**Acknowledgements** The author is indebted to Prof. Qing-Hu Hou at Nankai University for his help in finding recurrence relations and proof certificates concerning Lemmas 3.1 and 4.1.

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